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Form finding of tensile structures in Civil Engineering classes

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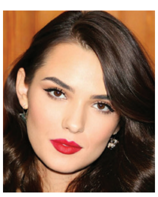

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Research Paper

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Form finding of tensile structures in Civil Engineering classes

In Structural Analysis 2, the third year's course of undergraduate studies at the Faculty of Civil Engineering, University of Zagreb, students acquire theoretical knowledge on the design of tensile structures (with emphasis on prestressed cable nets). Prior to calculation, to better understand the problem of form finding, students are assigned to build a physical model of tensile structure from elastic mesh fabric. Following, they apply the force density method to numerically determine the equilibrium position. To establish numerical model, Rhinoceros and Grasshopper (visual programming language within Rhinoceros) are used, while calculations are made in SageMath, an open-source mathematical software.

Key words:
undergraduate university study, Structural Analysis 2, form finding, physical model, numerical model

Prethodno priopćenje

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Traženje oblika vlačnih konstrukcija u nastavi na Građevinskom fakultetu

Na nastavi kolegija Građevna statika 2. na trećoj godini preddiplomskoga studija na Građevinskom fakultetu Sveučilišta u Zagrebu studenti niz godina usvajaju teorijsko znanje o oblikovanju vlačnih prednapetih konstrukcija (s naglaskom na mreže prednapetih kabela). Radi boljega razumijevanja problema traženja oblika, a primjereno preddiplomskom studiju, studenti su kolegija akademske godine 2019./2020. izrađivali fizički model vlačne konstrukcije od elastične mrežaste tkanine. Potom su primjenom metode gustoća sila numerički definirali ravnotežni položaj primjenom programa Rhinoceros i Grasshopper (vizualni programski jezik i okruženje u Rhinocerosu 3D) te SageMath.

Ključne riječi:
preddiplomski studij, kolegij Građevna statika 2., traženje oblika, fizički model, numerički model

Vorherige Mitteilung

Petra Gidak, Elizabeta Šamec, Krešimir Fresl, Jelena Vukadin

Formfindung von Zugstrukturen im Unterricht an der Fakultät für Bauingenieurwesen

Im Rahmen des Kollegiums Baustatik 2 des dritten Jahres des Diplomstudiums der Fakultät für Bauingenieurwesen der Universität Zagreb erwerben Studenten schon seit einigen Jahren theoretische Kenntnisse über die Formung von vorgespannten Zugkonstruktionen (mit Schwerpunkt auf vorgespannte Seilnetze). Um das Problem der Formfindung besser zu verstehen, und dabei im Rahmen der Möglichkeiten eines Diplomstudiums zu bleiben, haben Studenten des Kollegiums im Laufe des akademischen Jahres 2019/2020 das physikalische Modell einer Zugstruktur aus elastischem Netzgewebe hergestellt. Anschließend definierten sie unter Anwendung der Kraft-Dichte-Methode und mithilfe der Programme Rhinoceros und Grasshopper (visuelle Programmiersprache und Umgebung in Rhinoceros 3D) und SageMath numerisch die Gleichgewichtsposition.

Schlüsselwörter:
Diplomstudium, Baustatik 2, Formfindung, physikalisches Modell, numerisches Modell

1. Introduction

At the end of the winter semester of the academic year 2019/2020, students of the course Structural Analysis 2 (fifth semester of undergraduate study) at the Faculty of Civil Engineering, University of Zagreb were introduced to the problem of form finding of tensile structures by making a physical model. Although optimization of load-bearing capacity of static systems through shape alteration is applicable to all types of structures, finding the (equilibrium) shape in the case of tensile (flexible or suspended) structures is the main (and initial) task of the designer. Cables, as load-bearing elements of a prestressed cable net, are highly sensitive to the changes in position and direction of external load as they must define a net that forms an anticlastic surface (surface with negative curvature). To balance external loads, cables must change shape by activating tensile forces since considering their negligible flexural stiffness compressive forces and bending moments (consequently also shear forces) cannot occur in their cross sections. Therefore, the shape of the net and the level of prestressing are key parameters of the load-bearing capacity of prestressed cable nets (and tensile structures in general).

Chapter 2 briefly describes Frei Otto's contribution to form finding of tensile structures and their design in general. With Otto the modern age of prestressed tensile structures begins and, as it can be seen from the design solutions of load-bearing systems of considerable spans or floor plan dimensions (some of which are mentioned in subchapter 2.1), it continues today. Intention of this article is also to emphasize the potential of using tensile load-bearing systems in our country, since so far, their application has been limited (subchapter 2.2). The basic numerical problems and design solutions are explained in Chapter 3, while in Chapter 4 the search for optimal shapes is conducted in the same way as it was done in the past, manually, by building physical models. Further on, Chapter 5 describes the principles of form finding using numerical models, more specifically, the program code based on the force density method that is associated with modern graphic programs for better visual control and evaluation of shapes. Also, some future plans of the authors (related to education on form finding) are mentioned in the conclusion of the paper (Chapter 6).

2. Frei Otto – equilibrium virtuoso

It is impossible to write about tensile structures without mentioning the main "culprit" for redefining the design principles and better understanding of tensile systems - it is, of course, Frei Otto (1925-2015). With the establishment of the Institute for Light Structures at the University of Stuttgart (Institut für Leichte Flächentragwerke, Universität Stuttgart), led by Otto, systematic study of light shape dependent structures began (Figure 1).

Before computer age, due to the limits in application of analytical methods, the problem of form finding was approached through detecting the coordinates of reference points from physical models, mostly based on minimal surfaces (even a device was developed for "production" of soap membranes as natural minimal surfaces). The coordinates obtained by using photogrammetry did not meet the required accuracy which was highly important considering the sensitivity of the equilibrium shape to the displacements of the structure's points, especially for large spans. And of course, detecting the intensity of prestressing forces was out of question.

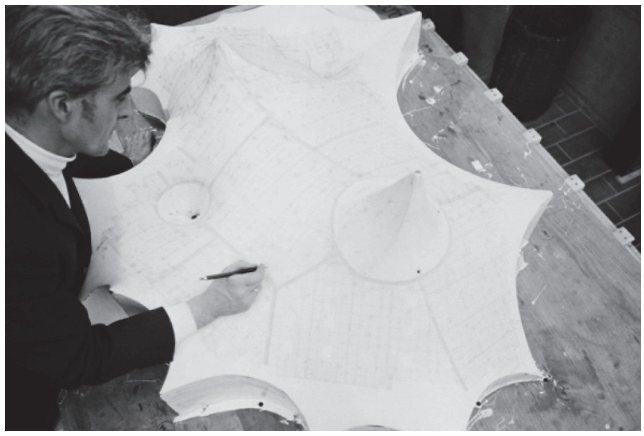


Figure 1. Frei Otto working on the roof model for the swimming pool in Olympiapark Munich (https://zkm.de)

During the design of the Munich Olympic Park facilities – the roof of the pool and athletic stadium, Otto and his project team sought the help of mathematician and computer scientist Hans-Jörg Schek (b. 1940) and geodesy and civil engineer Klaus Linkwitz (1927-2017). Their collaboration resulted in the first computer program for form finding of tensile structures based on a newly formulated method for solving systems of nonlinear equations, more precisely, systems that are formed by equilibrium equations of prestressed cables nets nodes, that are highly nonlinear. Schek and Linkwitz called the method the force density method (more on this in Chapter 3) [1, 2]. The basic idea of the method even today remains the starting point in development of new methods for form finding of shape-dependent structures (such as prestressed cable nets, membrane structures [3], single-layer reticulated shells [4], tensegrity systems [5], bending active structures [6], etc.).

2.1. Freeing the form

Two structures can be considered as forerunners of Otto's Munich project: the David S. Ingalls Rink Hockey Hall at Yale University by architect Eero Saarinen (1958), and the Yoyogi Gymnasium in Tokyo by Kenzo Tange (formerly the Olympic Water Sports Hall, 1964). Both roofs consist of prestressed cables, whose form needed to be found even though we cannot consider it a true tensile net. Otto's roofs in the Olympic Park (especially the roof of the Olympic Stadium) are the first permanent nets of long-span prestressed cables, build thanks

to the development of already mentioned force density method. With their construction (only half a century ago), begins the rapid development of tensile prestressed structures, and tensile load-bearing systems reveal their irrefutable advantages (especially for large span solutions), but also some drawbacks (significant dimensions and (or) weights of supporting structure for acceptance of especially large reactive tensile forces).
The list of prominent tensile structures must inevitably include the Hajj terminal of Jeddah International Airport (Saudi Arabia, built in 1980) as the largest roof area in the world (465,000 m2), the roof of King Fahd Stadium in Riyadh (Saudi Arabia) from 1987 and the Denver International Airport terminal building (USA, from 1994.). On Figures 2 and 3 more recent achievements are presented. In Figure 2 the Khan Shatyr entertainment center in Astana whose construction was completed at the beginning of the 21st century and whose dimensions are still impressive: the highest point of the prestressed cable net is 90 m (which makes it the tallest tensile structure), while the elliptical floor plan measures 200 m x 195 m.

Slika 2. Amusement center Khan Shatyr in Kazahstan, 2010. (CC BY-SA 4.0)

In Figure 3, the bicycle arena built for the London Olympics (2012) is shown, as an example of the evidently elegant, also undeniably efficiency tensile structure.

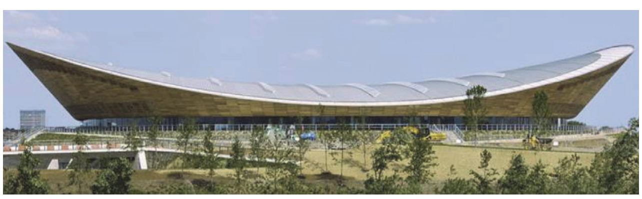


Figure 3. Velodrome (cycling arena) in London, Great Britain, 2011 (archdaily.com)

It is a double net of prestressed cables of 12,500 m^2 (the largest spans in the two vertical directions are 131 m and 119 m). The shape of the cable net is defined with help of the conditioned correlations between prestressing forces. The supporting structure consists of a metal ring (spatial truss) and concrete pillars (largest dimensions 3.2 m x 0.75 m) as well as a concrete slab (25 cm thick) at ground level.
Today, the principles of form finding of prestressed tensile structures are also applied to the design of more efficient conventional structural elements or to the design of single-layer reticulated shells using the analogy of equilibrium forms in tension and compression [4] (hanging cloth analogy). At ETH Zurich, a group of researchers led by Phillip Block, the Block Research Group (BRG), developed a ribbed stiffened funicular floor system that is 70 % lighter than typical floor slab that is 25 to 30 cm thick. The compression-only shell is designed for self-weight and dead load, and with optimized amount and arrangement of thin ribs for the remaining live load combinations the need for conventional rebar is entirely taken away [7].

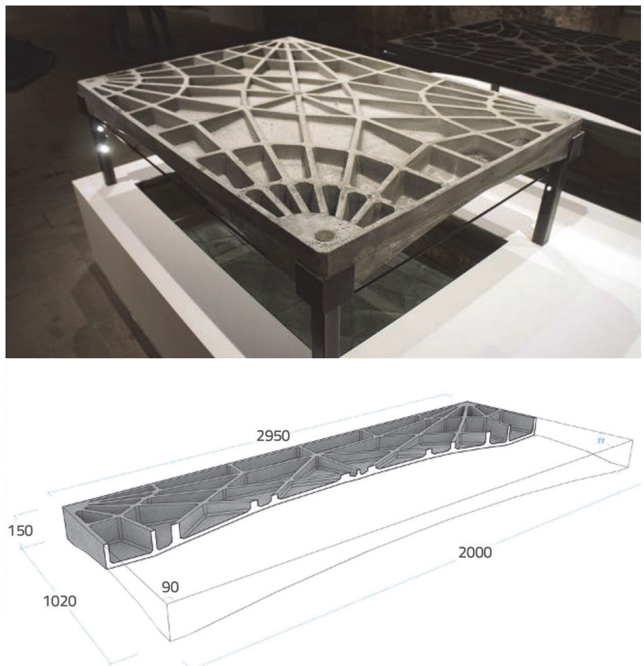


Figure 4. Prototype of thin concrete slab (up). Perspective section of the prototype with units in mm (down), [7]

2.2. Where does Croatia stands?

2.2.1. Form finding limbo at the Department of Engineering Mechanics

At the Department of Engineering Mechanics on the Faculty of Civil Engineering in Zagreb, the systematic research on form finding problem began in 1997 with the publication of papers [8, 9] by prof. emer. Josip Dvornik and prof. dr. sc. Damir Lazarević. More than 20 years later, in their latest work [10] (together with prof. emer. Nenad Bičanić) they did not omit prestressed tensile structures (pp. 396–425) either. However, [8, 9] were only a consequence of a series of "tensile topics" that preoccupied the authors since the 1980s. Dvornik's contribution (with associates Ramiz Fejzo and prof. dr. sc. Joško Ožbolt) to the Poljud's

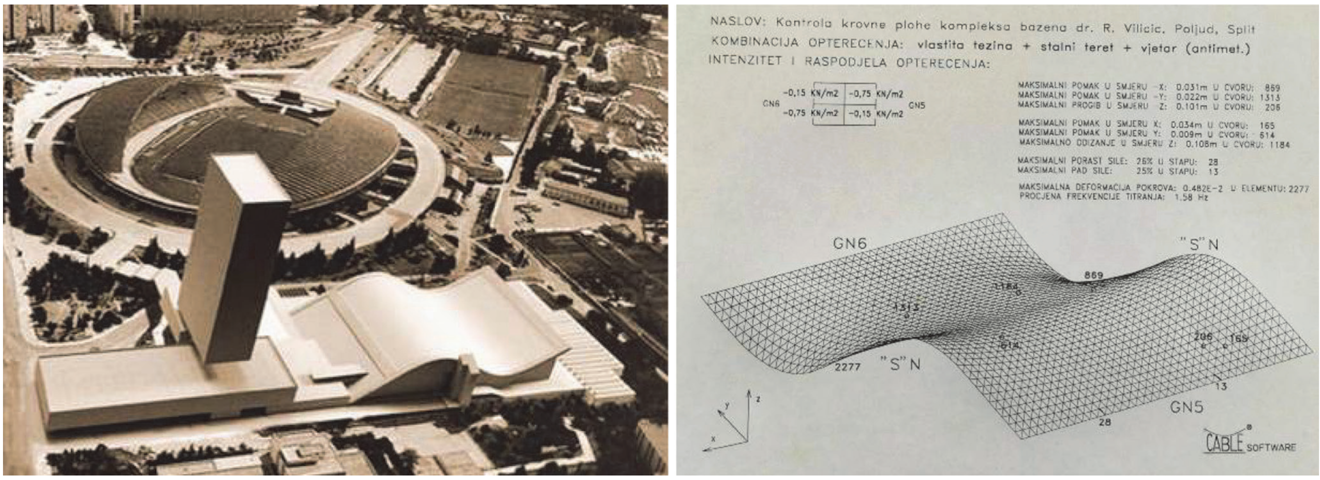


Figure 5. Swimming pool hall at Poljud, Split, 1979 (left). Numerical model of pool's roof structure (right).

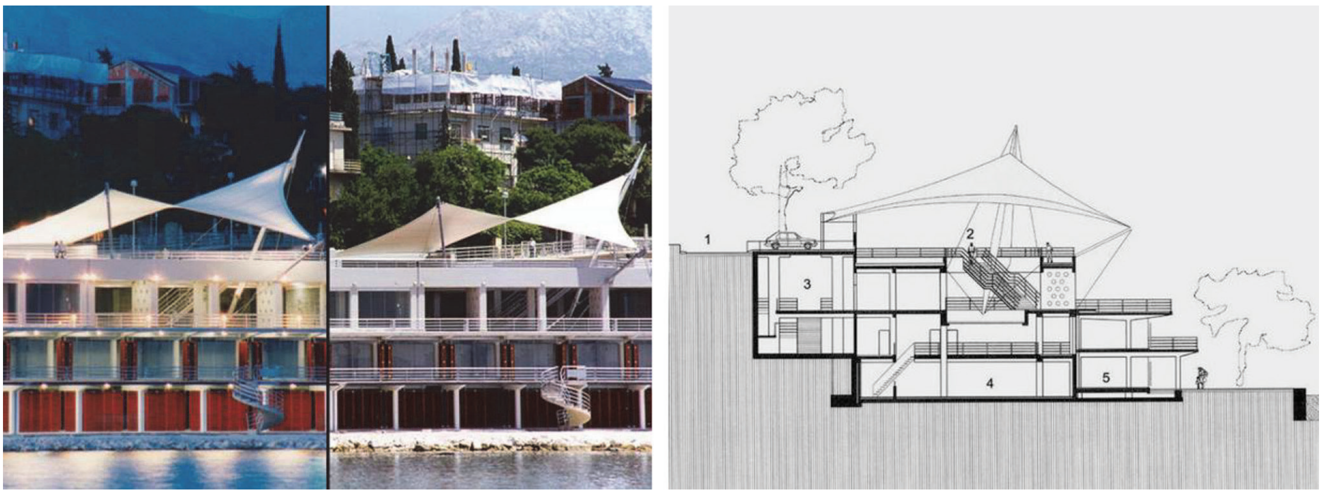


Figure 6. AKB Beach Complex Bačvice, Split (1998)

swimming pool roof repair, (Split in 1979, Figure 5 left) could be considered the first official, but indirect, connection between the Faculty and tensile load-bearing systems in Croatia (if we do not count discussions at the corridors and in Department offices). We say, "indirect connection", because Dvornik and colleagues did not "repair" the load-bearing system of the pool's roof (a prestressed cable net), but its complementary construction. Recognizing the problem of excessive span and too small cross-section of the main S-beams (with insufficient stiffness to take over the horizontal forces from the prestressed cable net), they proposed to introduce additional columns and struts into the complementary structure, and to prestress the transversal beam from the outside.

In the early 1990s, the research of nonlinear behavior of tensile structures at the Department resulted in the development of a computer program CABLE for determining the shape of tensile and other light structures (among others, tension-compression analogy was applied as well as static and seismic analysis). CABLE was a direct consequence of Lazarević's diploma thesis, where he already then envisioned its further development into an interactive program where changes in input data are directly visible on the result in equilibrium. CABLE was later used to check the shape and load-bearing capacity of the roof structure on Poljud (Figure 5 right).

Furthermore, the program was used to design a canopy on Bačvice beach in Split in 1998 (Figure 6). Unfortunately, CABLE is lost in the black holes of Departments shelves and floppy disks, while the idea of an interactive computer program is realized only today with the dissertation of the co-author of this paper, Elizabeta Šamec. Nevertheless, from CABLE to the present day, the form finding of prestressed tensile structures was also the topic of dissertations of the other two co-authors of this paper [11, 12] and several final and diploma theses, as well as papers, where we can point out [13].

Although this is only a conceptual solution, we certainly want to draw attention to the project of the canopy of the bus terminals at the Čilipi Airport (Figure 7). The basic idea of the authors, Dvornik and Lazarević, came from the desire to break the distinct linearity and regularity of the given floor plan. They also wanted to avoid the typical reinforced concrete and steel

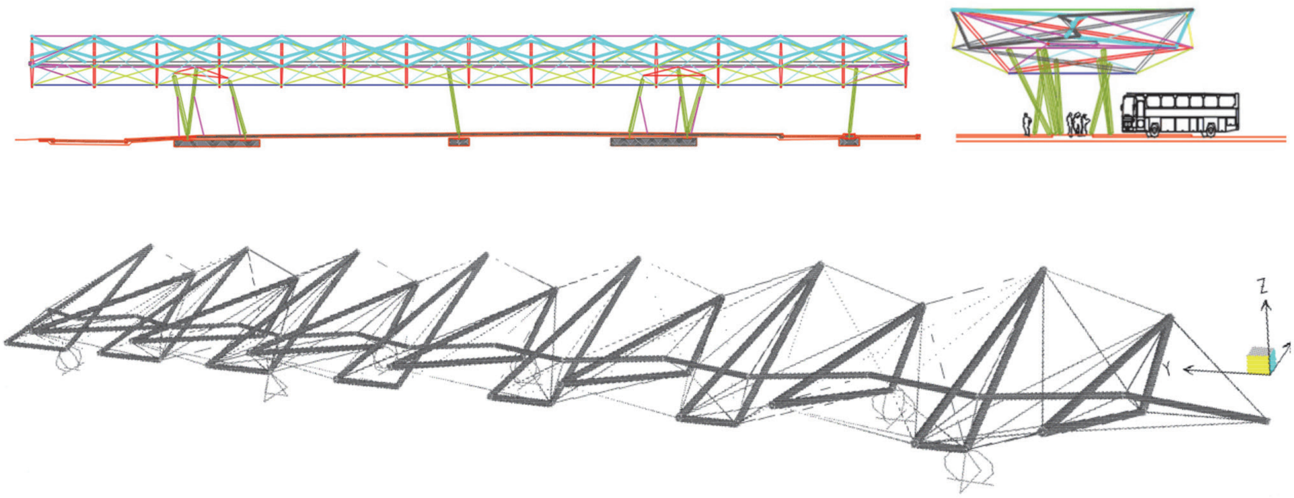


Figure 7. Conceptual solution for canopy at Čilipi Airport near Dubrovnik (2005)

solutions that appear heavy and bulky and would suffocate Mediterranean atmosphere present in Dubrovnik's area. It was necessary to design a light and playful construction, and the design duo wanted to add uniqueness by choosing a very modern load-bearing system - tensegrity construction (that would be the only such structure in Croatia). The project stalled in the conceptual design phase, so no detailed static calculation was carried out. The shape of the basic modulus of tensegrity structure was determined for prestressed, the dimensions of ropes and compression rods were selected, and a simple calculation for the action of wind was performed according to forces taken from regulations. More details can be found in [14].

2.2.2. Some tensile canopies in Croatia

If we look outside the Department's work, from the time of construction of the Poljud swimming pool to the present day prestressed tensile load-bearing systems served as effective architectural solutions for several canopies. Figure 8 shows some of them, while following are also known to us: the canopy of the west tribune of the athletic stadium Mladost in Zagreb, the canopy at the tram turnaround Borongaj in Zagreb, the canopy of the railway station Karlovac - center, the canopy of the bus station in Zaprešić and the canopy in front of Bočarski dom in Zagreb. We are sure that there are other tensile prestressed canopies in our country, however, the 64 m long span of the prestressed cable net for the pool on Poljud has not been surpassed by the same or similar load-bearing system. More precisely, as far as we know, in Croatia there is no other tensile structure of similar span (which we could consider large).

Photos: we thank the Architectural Bureau Ante Kuzmanić (Figures 5, 6 and 7), Robert Kriković (Figure 8 left), and VV projekt d.o.o. (Figure 8 right).

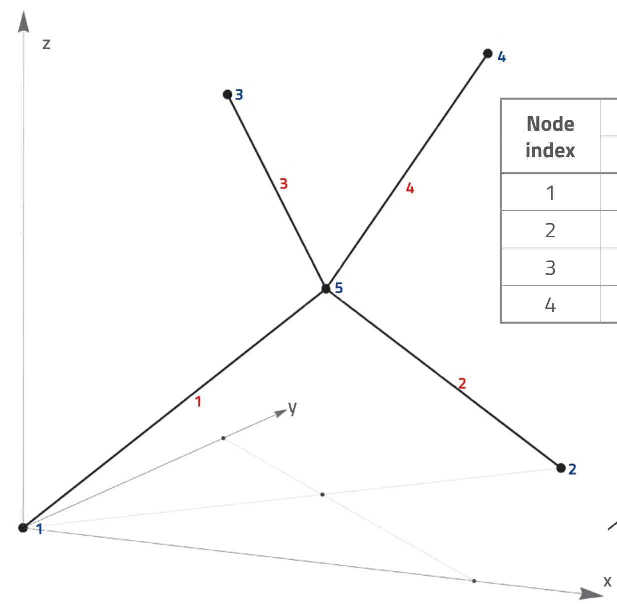
3. About the problem of Newton and Raphson and the solution of Schek and Linkwitz

In the form finding process it is assumed that cables are completely flexible. Furthermore, cable axes pass by each other due to the (small but existing) cross section dimension. However, in calculations we can suppose that they intersect. The points of intersection are called nodes, and we distinguish between

Figure 8. Zadar Airport (2012) and bus terminal at Split Airport (2019)

free (coordinates whose equilibrium position is unknown) and bearing nodes - anchors (their coordinates are the input data in the form finding process). The parts of the cable between the nodes are defined as the elements of the cable net, which due to the high value of tensile force and the assumption of complete flexibility can be considered as parts of lines (Figure 9). Due to the very small value of weight in relation to the value of the prestressing force, the dead weight can be neglected, as well as the external loads, but only in the form finding phase. Namely, after determining the equilibrium shape of the net of prestressed cables, a static calculation is performed in which external loads (with magnitude, direction, and place of action) and their combinations are added. The equilibrium shape obtained in the form finding phase will not change significantly in the static calculation (the displacements of the net nodes will remain small enough). Specifically, if, for example under the action of wind, the shape changes significantly (or "relaxed" places appear), the form finding process should be repeated and a new equilibrium shape should be found with increased values of prestress forces or by changing position of anchor nodes.

Figure 9 shows a net of two prestressed cables that intersect in a node marked with index 5. The coordinates of the other nodes are known, and they are the input data for the process of finding the equilibrium position of free node 5. By isolating the node 5, that is connected to elements indexed from 1 to 4, three equilibrium conditions are set. Even though node 5 is a part of space structure, there are only three independent equilibrium conditions (as already mentioned, in the cross sections of the cables bending moments cannot occur due to negligible flexural stiffness of the cable). Therefore, the intersections of the cables can be considered as hinges and three equilibrium equations are:



Node index	Node coordinates		
	x [m]	y [m]	x [m]
1	0.0	0.0	0.0
2	2.0	2.0	0.0
3	0.0	2.0	2.0
4	2.0	0.0	2.0

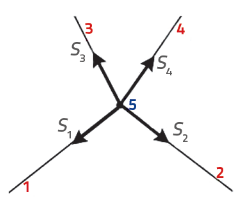


Figure 9. Example 1 – net consisting of two cables

$$
\begin{aligned}
&S_1 \cdot \cos\alpha_1 + S_2 \cdot \cos\alpha_2 + S_3 \cdot \cos\alpha_3 + S_4 \cdot \cos\alpha_4 = 0 \\
&S_1 \cdot \cos\beta_1 + S_2 \cdot \cos\beta_2 + S_3 \cdot \cos\beta_3 + S_4 \cdot \cos\beta_4 = 0 \\
&S_1 \cdot \cos\gamma_1 + S_2 \cdot \cos\gamma_2 + S_3 \cdot \cos\gamma_3 + S_4 \cdot \cos\gamma_4 = 0
\end{aligned}
\tag{1}
$$

where

$$
\cos\alpha_i = \frac{x_i - x_j}{\ell_i}, \quad \cos\alpha_i = \frac{y_i - y_j}{\ell_i}, \quad \cos\alpha_i = \frac{z_i - z_j}{\ell_i} \tag{2a}
$$

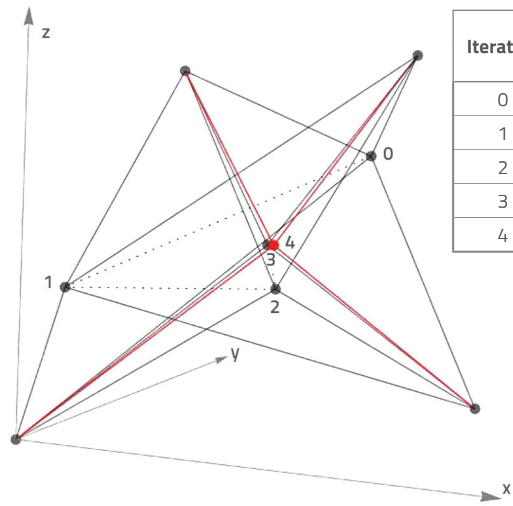
are the cosines of the angles that the individual elements close with the x, y, and z coordinate axes, and

$$
\ell_i = \sqrt{\left(x_i - x_j\right)^2 + \left(y_i - y_j\right)^2 + \left(z_i - z_j\right)^2} \tag{2b}
$$

is the length of the individual element. Each equation in (1) is the sum of the projections of the (internal tensile) forces of all the elements that meet at node 5 on a given coordinate axis. The (three) coordinates of the free node (x_5, y_5, z_5) and the values of the forces (S_1, S_2, S_3, S_4) in the four cables that meet in that node are unknown. Thus, without considering additional assumptions (such as, for example, the equality of force values in all net elements), system (1) cannot be solved. Since node 5 is not loaded by external load, at least one of the four tensile forces in the node must have an "upward" direction (this can illustratively explain the double curvature of the equilibrium surface of tensile structures). Furthermore, the system of equations (1) is obviously nonlinear. When applying a standard iterative method for solving systems of nonlinear equations, such as the Newton – Raphson method, convergence depends on the choice of the initial position of the free node (ignoring the fact that this is an example whose solution can be easily guessed due to symmetry).

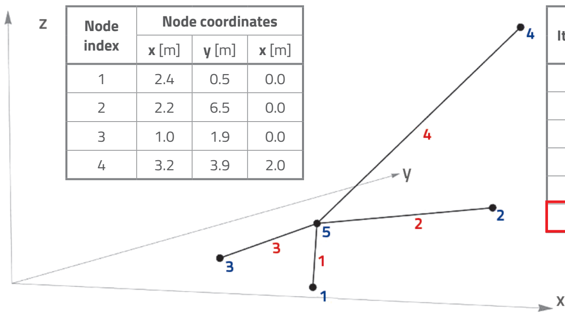
The Figure 10 shows the positions of the free node during iterations by the Newton-Raphson method with (random) selection of the initial solution (x_5^0, y_5^0, z_5^0) = (1.0, 3.0, 1.5). Furthermore, in the same figure, the dotted line indicates the path of the free node from the initial position to the arrival in the equilibrium position shown in red (with accuracy according to the Euclidean norm). However, with the increasing complexity of the prestressed cable net, guessing a good starting position is not an easy task. For example, if the coordinates of the anchor points in the example at Figure 9 change, and initial position selection is (accidentally) bad, the Newton-Raphson method can easily diverge (Figure 11) [15].

Therefore, the system of nonlinear equations (1) is solved by methods specific to the problem of form finding, where the force density method by Schek and Linkwitz is used in the course Structural Analysis 2 [1, 2].



Iteration	Free node coordinates		
	x [m]	y [m]	x [m]
0	1.0	3.0	1.5
1	0.56103	-0.5870	0.87907
2	0.8749	1.4292	0.70344
3	0.9596	1.0102	0.9977
4	0.9999	1.0000	0.9998

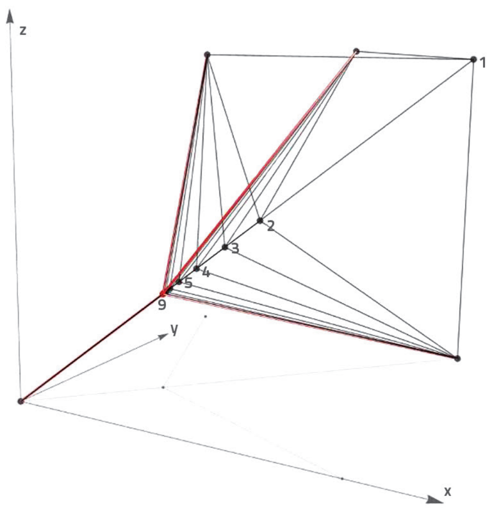
Figure 10. Position of node 5 (example 1) in each iteration step of Newton-Raphson



Node index	Node coordinates		
	x [m]	y [m]	x [m]
1	2.4	0.5	0.0
2	2.2	6.5	0.0
3	1.0	1.9	0.0
4	3.2	3.9	2.0

Iteration	Free node coordinates		
	x [m]	y [m]	x [m]
0	0.5	2.0	1.5
1	3.4463	2.6434	-1.5567
2	0.4536	2.3536	2.7852
3	4.9870	3.4094	-3.4607
4	-8.0319	1.7899	14.7863
5	539.081	74.4121	-747.03

Figure 11. Example 2 – coordinates of anchor points and node 5 by iteration steps of Newton-Raphson



Iteration	Free node coordinates		
	x [m]	y [m]	x [m]
1	2.0	2.0	2.0
2	1.0	1.0	1.0
3	0.8453	0.8453	0.8453
4	0.7226	0.7226	0.7226
5	0.6479	0.6479	0.6479
6	0.6096	0.6096	0.6096
7	0.5917	0.5917	0.5917
8	0.5836	0.5836	0.5836
9	0.5801	0.5801	0.5801

Figure 12. Position of node 5 (example 1) in each step of iterative application of force density method

If a factor, whose value represents a (new) input parameter, is entered in system (1), instead of the ratio of the value of the force in the element and its length, the nonlinear system of equilibrium equations becomes linear. Moreover, the whole system disintegrates into three independent equations:

$$\begin{aligned} &\sum_{i=1}^{4} q_i \cdot \left(x_i - x_5\right) = 0, \\ &\sum_{i=1}^{4} q_i \cdot \left(y_i - y_5\right) = 0, \\ &\sum_{i=1}^{4} q_i \cdot \left(z_i - z_5\right) = 0, \end{aligned} \tag{3}$$

where $q_i = S_i / \ell_i$, is named force density. The solution of the system are coordinates of a free node with index 5. Expression (3) is a system of linear equations that can be solved directly (so the initial position of the free node does not need to be assumed). By including the coordinates of anchors, from Figure 9, and by considering the (trivial) unit value of the force densities of all elements ($q_1 = q_2 = q_3 = q_4 = 1$), it is easy to determine the solution: (x_5, y_5, z_5) = (1.0, 1.0, 1.0).

The force density method can be applied iteratively by introducing constraints in the form of desired lengths (all or some) of net elements or values of forces in them (both conditions cannot be set simultaneously for the same net element). In this case, the force density of the constrained element, in some step of the iterative calculation, is determined according to the expression $q_i^k = S_i^k / \ell_i$ or $q_i^k = S_i / \ell_i^k$, where $\overline{\ell}_i$ is wanted element length with index *i*, and $\overline{S}_i$ is element's wanted force value, also with index *i*. For the example in Figure 9, the equilibrium position of the free node will be determined by iterative application of the force density, where on the length of element 1 kinematic constrain is set: 1 : $\overline{\ell}_1$ = 1.0. Figure 12 shows the path of the free node during the iterative calculation. Net nodes are in equilibrium position in each of nine steps from Figure 12, but only the position of the cables marked in red satisfies the kinematic constraint of element 1. By comparing the free node paths in Figures 10 and 12, we can indisputably see the nature of the Newton-Raphson method, that wanders towards the solution, while by using force density method, a conditioned equilibrium position of the node is reached by translation along a much smoother curve (in this example, due to the symmetry the curve is a line).

Due to the nonlinear nature of the form finding problem and the fact that the form of the static system, on which static calculations are to be performed, is unknown in advance (as opposed to static calculations in most Civil Engineering problems), teaching this topic is often very challenging. Therefore, experiencing form finding by building a physical model serves as a good introduction to basics of designing tensile structures.

4. Physical model – form finding by hands

Learning on form finding is inspired by the works of "old" masters and visionaries Antonio Gaudí and Frei Otto, pioneers of structures whose shape is not known in advance - it is determined by equilibrium conditions. The design problem was therefore turned into shape exploration by building physical model, as it was done in the past, and on the other hand by using modern tools for numerical calculation. The intention of both approaches was to encourage students to explore these likable architectural and construction solutions.

The basic rule when exploring shape by building a physical model is to avoid creasing (relaxation) of the fabric i.e. to find a smooth surface for selected boundary conditions, the one that is anticlastic by nature (it should be noted that the choice of fabric was such that creation of cutting pattern was not necessary).

Students were divided into groups consisting of four to five members since cooperation and "multiple pairs of hands" were needed to successfully create a model. Each group found inspiration in the existing (or former) tensile flexible structure or its model and, in consultations with lecturers, they identified the necessary input data - the position of anchor points and how the membrane rests on a complementary (bearing) structure. The floor plan positions of all anchors are then sketched on a thick cardboard base on which afterwards the fabric is placed and fixed in the initial position (in the plane of the base) (Figure 13).

The minimum number of bearing points is four, where at least one must be elevated out of the plane of the other three. This ensures a saddle form i.e. equilibrium position is described by a surface of double curvature. The next step is to raise the membrane from initial ground plane (Figure 14), where the raised anchor can be supported on the top of the pillar, connected with the cable that stretches from the pillar or the fabric can be continuously attached to the arch.

The high anchors can be additionally secured by sewing the fabric around the tops of the pillars or along the arches. Pillars and arches are parts of a complementary structure that in a real static system takes on significant bearing reactions. Therefore, when model is made, the pillars are stabilized with additional cables, and the foundations are simulated with pieces of styrofoam that is glued or sewn to the cardboard.

After achieving a smooth surface, the excess fabric that folds or has too little tension needs to be cut out (Fig. 15).

For the anticlastic surface without loose places to be formed, the procedure sometimes requires "manual iterations" of

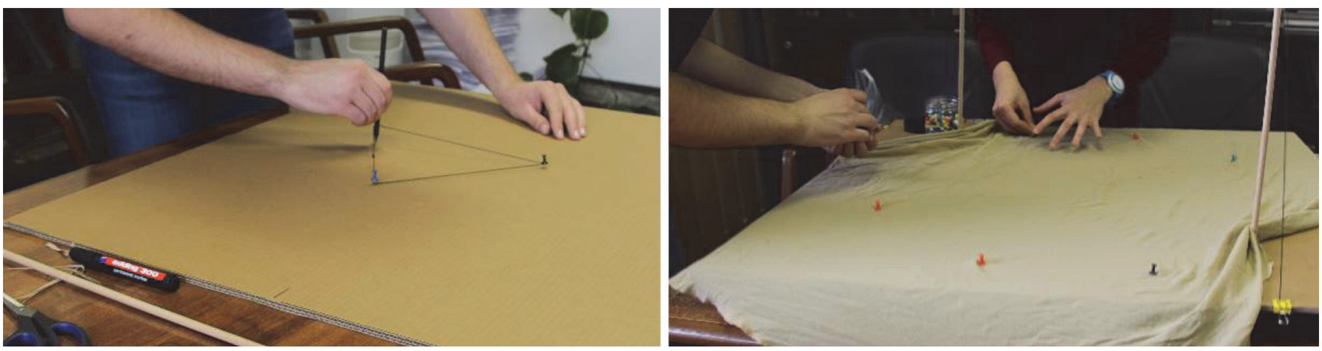


Figure 13. Building physical model of tensile structure – marking anchor points and initial position of fabric

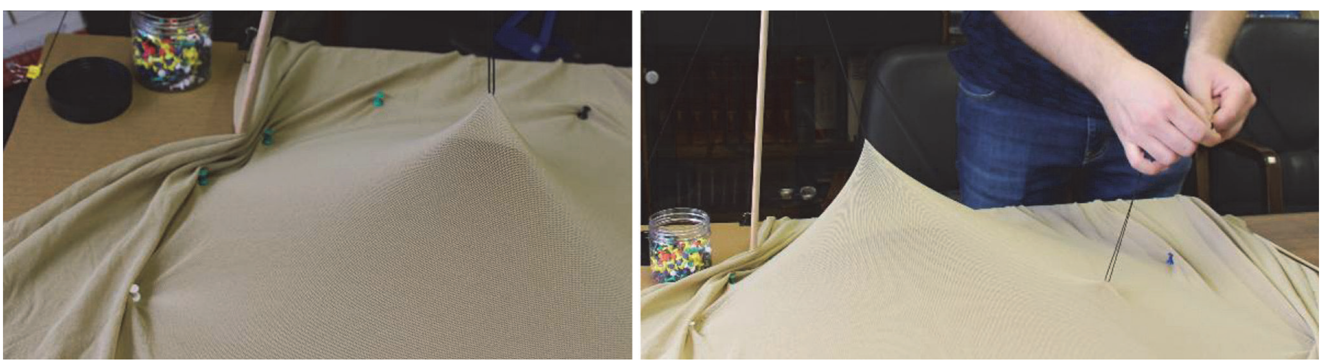


Figure 14. Building physical model of tensile structure – raising high anchors

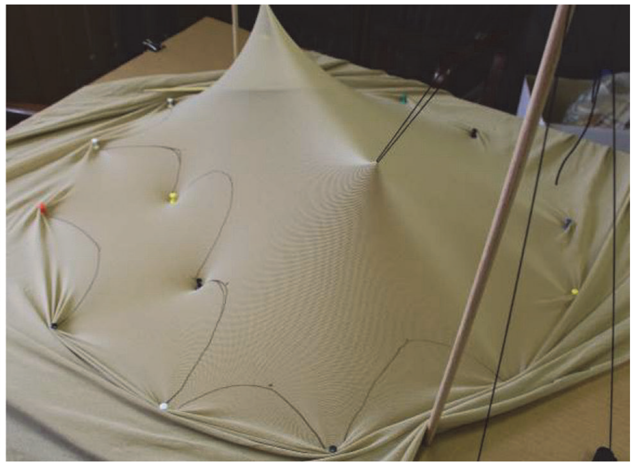


Figure 15. Building physical model of tensile structure – drawing edge curves

anchors positions or the shape of the supporting structure. With the same problems one needs to face when conducting form finding by using a numerical model.

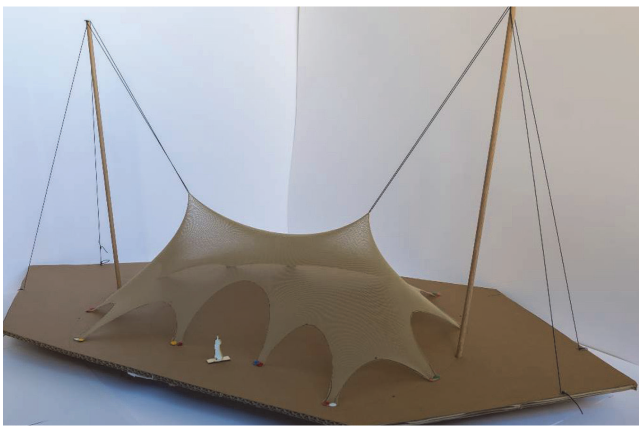


Figure 16. Building physical model of tensile structure – final shape (authors: I. Ladavac Jančin, E. Šamec and P. Gidak)

5. Numerical model of tensile structure

5.1. Selection of numerical model

After finding the shape using a physical model, the students were encouraged to make a numerical model. Since fabric that did not need creation of cutting pattern was used as the load-bearing system, it cannot be claimed that the obtained physical models fully correspond to the shapes of the membrane structures. On the other hand, the models cannot be declared as prestressed cable constructions either. However, for their form finding, procedure for cable nets can be used since in form finding of membrane, a surface can be discretized by the cable elements (consciously accepting a certain level of inaccuracy of the equilibrium shape). In practice, an analogy with a cable net is often used to find the shape of membrane, in which the cables are aligned with the warp and weft directions of the threads. The analogy can be applied since the warp and weft directions are mutually independent, and therefore the stress in the warp direction is caused solely by the modulus of elasticity and deformation in that direction. Cable elements represent polygonal elements that are formed by discretizing the membrane into strips of certain widths, in both directions. Something about the influence of discretization density on the accuracy of the approximation can be found in [16]. Once the shape is found by one of the existing methods (such as force density or dynamic relaxation methods or one of the stiffness matrix methods [17]), the obtained values of forces in cables represent the values of stress results acting on defined membrane strip widths. Commercial software (e.g. *EASY* [18]) often use the described methodology.

To find the shape of the cable net in the class, the (previously developed) computer source code based on the force density method [13, 19] was used, where the code is developed in mathematical software *SageMath* [20]. To make the generation of input data easier (and to assure easy graphical verification of the input data), the 3D CAD program *Rhinoceros* with the corresponding plug-in for visual programming *Grasshopper* [21] is used to assign the net topology and boundary conditions. Form finding tools (e.g. [22]), whether stand-alone or as additions to the *Rhino/Grasshopper*, are generally based on the dynamic relaxation method or the one-step force density method to find the equilibrium shape. Therefore, creation of our own materials was necessary to compensate the lack (public unavailability) of tools for constrained form finding based on force density method.

5.2. Defining topology and boundary conditions

It was mentioned earlier that the force density method does not require the selection of initial coordinates of the net nodes but assumes that the ratio of the value of the force and the length of the cable element is predetermined. Therefore, to find the shape, it is enough to know the topology of the net (connection of nodes), to determine which nodes are bearing and set them on the desired positions and to assign force density values. To enable students to interactively set the input parameters (and to verify them in a graphical manner), a so-called script (command file, program add-on) was prepared in the *Grasshopper* (GH) visual programming language. As shown in Figure 18, by using such script students can easily associate their initial net definition drawn in 3D CAD tool *Rhinoceros* (in this case for the model from Figure 17) and interactively manipulate with the position of anchor points, Figure 19. In that case, for the definition of net topology in GH script it is necessary to set initial coordinates of net nodes (usually in floor plan, primarily for topology display).

To establish a net of elements and nodes from the selected cables, without the need to draw many elements, the *Net from Lines* component is used, which can be found in Heteroptera

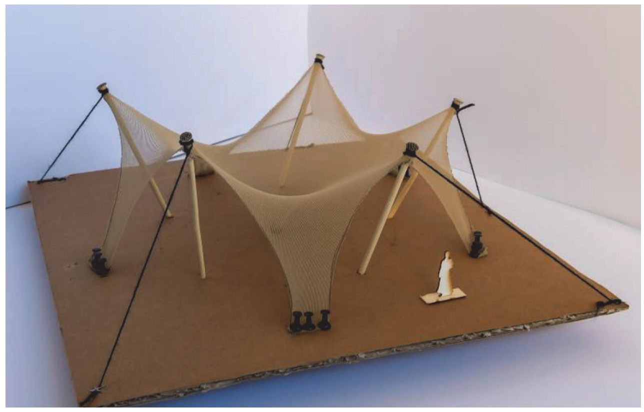


Figure 17. Physical model 1 (authors: K. Dropulja, T. Dučić, M.-P. Dušati, A. Dvorski, R. Filipović)

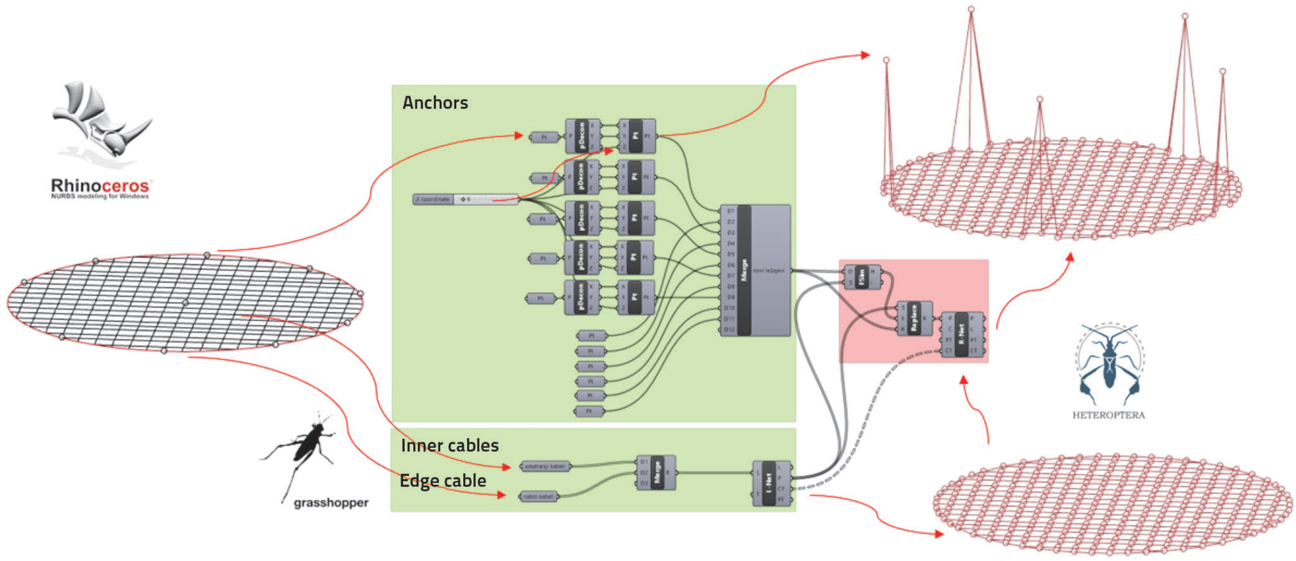


Figure 18. Assignment of drawn geometry and interactive manipulation of anchor points

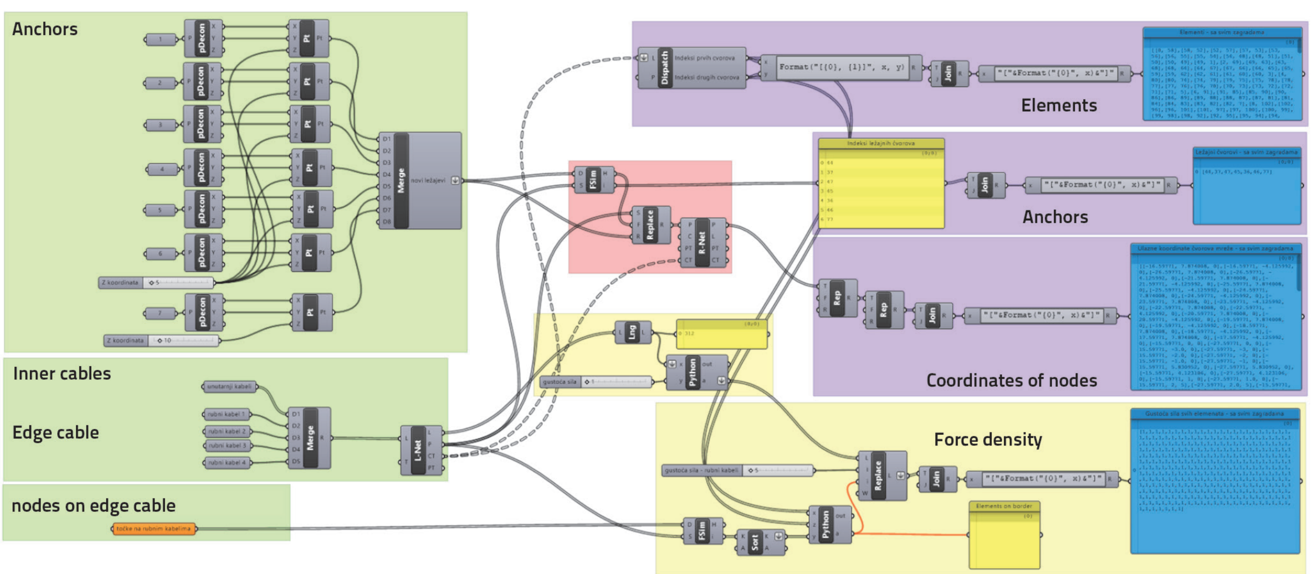


Figure 19. GH script for assignment of CAD geometry and preparation of input data for *SageMath*

[23], a GH plug-in. Nodes, which are defined as anchors, can be moved around interactively, outside the initial xy (floor) plane, and they can be changed, or new ones can be added. By using the *Rebuild Net* component, from the same plug-in, the net is every time updated according to the changes in the coordinates of anchor points. In this way, the output data necessary for form finding with the force density method is directly refreshed (blue panels in Figure 19). The described method of manipulating input data is clear and simple and due to its visual nature, it is easy to spot potential errors.

In this paper, in all examples, a unit value of force density was used for internal elements, while for the elements of edge cables this value was increased depending on the desired tension of certain or all edge cables (yellow group of GH components in Figures 19 and 22).

Once the GH script is defined, the necessary input data for any net can be quickly obtained, with minor modifications depending on the number of anchor nodes.

5.3. Numerical form finding

The output data of the GH script (contents of the blue rectangles) is used as the input data (type list) for the form finding function in *SageMath*. The declaration of function for finding the equilibrium shape is:

```
def FDM (nodes, elems, supports, qs)
```

where `nodes` is the list of coordinate nodes, `elems` the list of elements, `supports` the list of anchor's indexes and `qs` the list of force densities. The function returns coordinates of nodes of the net in equilibrium position, that can be loaded into a 3D CAD tool and by using known net topology the new shape can be displayed (source code in *SageMath* also contains a function for graphical representation of the equilibrium net, but for more complex manipulations user's intervention in the program code is necessary). The equilibrium position of model 1, found using the given function, is shown in Figure 20.

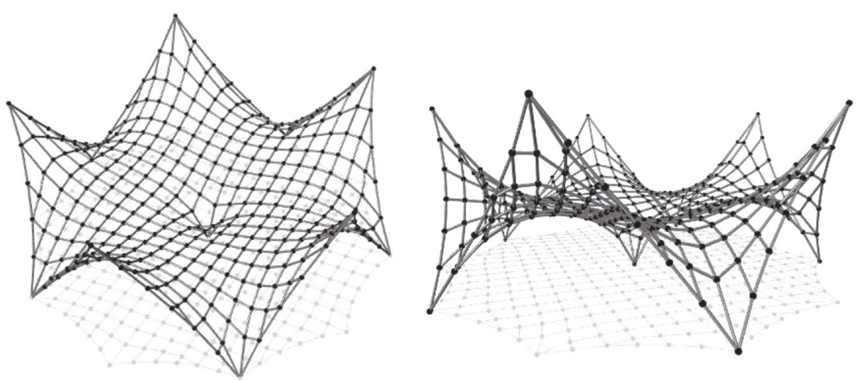


Figure 20. Model 3 –form finding solution

5.4. Constraints on lengths and force values in elements

Finding an equilibrium position does not guarantee a solution that is also structurally satisfactory, i.e. applicable. Although changes in force densities of individual cables can have some effect on the shape (the value of the force in the element usually increases with increasing force density), the lengths of the elements connected to the edge cable or fixed edge are then often inappropriate and nodes undesirably group on small intervals. This can be circumvented by setting additional initial conditions (constraints) such as desired element lengths or force values in individual net elements or along the entire cable.

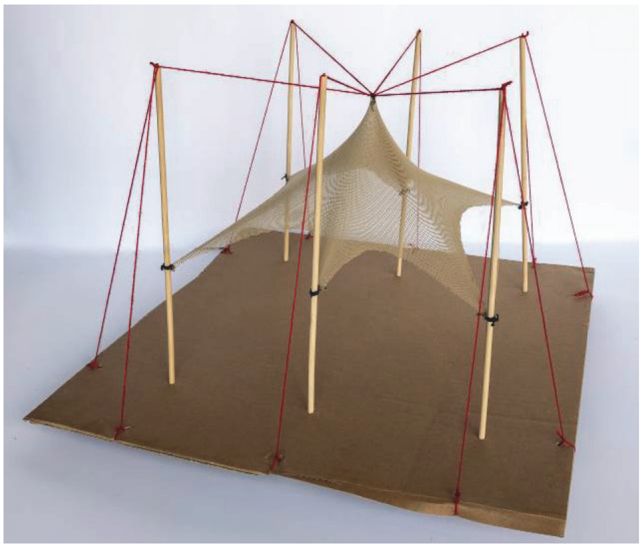


Figure 21. Physical model 2 (authors: F. Pigac, M. Pintarić, K. Ramljak, M. Repić, M. Vujeva)

In the numerical model of the structure with a high (central) anchor point from Figure 21, different force densities are assigned to the edge cables, higher in the y direction (Figure 22), to get the shape as close as possible to the physical model.

The obtained equilibrium shape (Figure 25 left), although similar to the original model, is not satisfactory due to the large lengths of the elements around the high point. Therefore, the lengths constraints are set for the elements connected to that point. To find the indexes of these elements more easily, the program code (GH component in Figure 23) written in Python was added to the script (Figure 23).

The form finding with additional constraints is performed by the function in Figure 24 where `fcs` and `lcs` are lists of

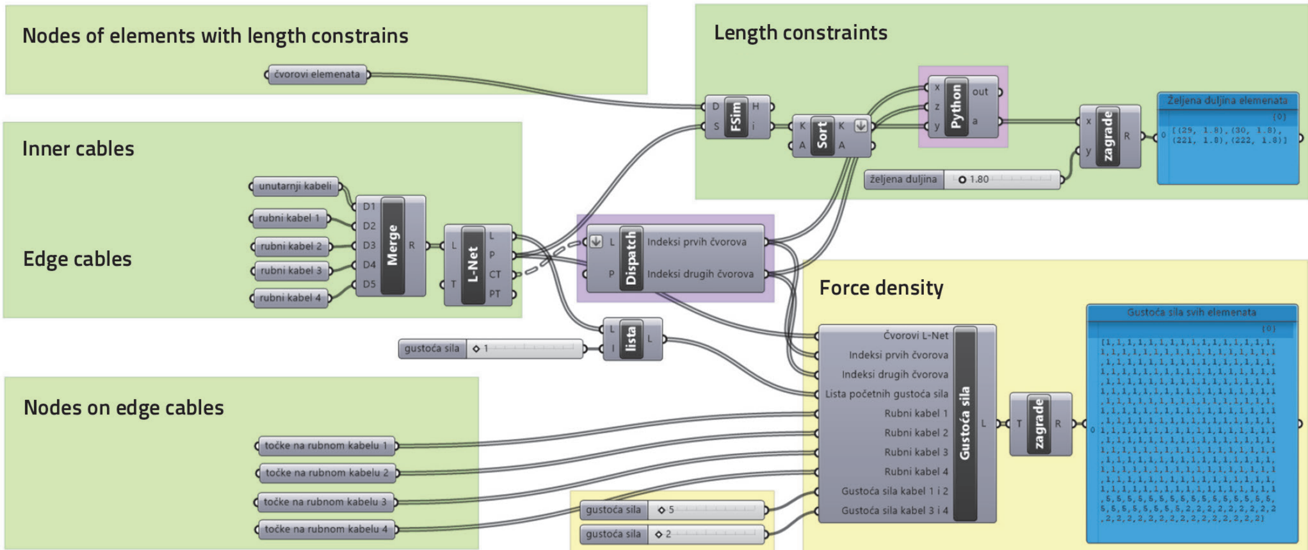


Figure 22. GH script for assignment of different values of force densities in edge cables and length constraints

pairs of element indexes and required values of forces or element lengths given by GH script in the top blue panel (Figure 22).

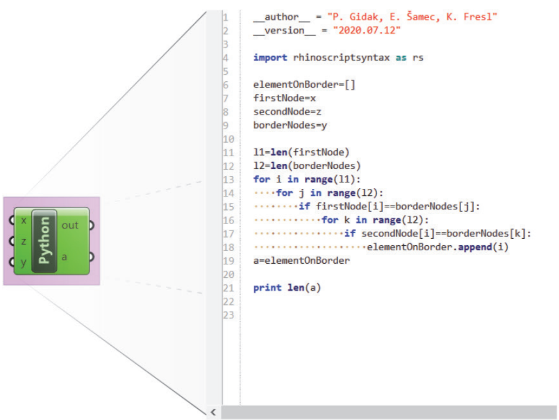


Figure 23. Custom *GhPythonScript* component for tracing indexes of selected elements

The `multistepFDM` function enables usage of iterative application of force density method in a given number of steps (`steps`), where in each step the force density value changes as a ratio of the required force value and the length achieved in the previous step, or as a ratio of the force achieved in the previous step and the required length [13], which ultimately leads to the fulfilment of set constraint (sometimes the set constraint cannot be met for numerical or physical reasons).

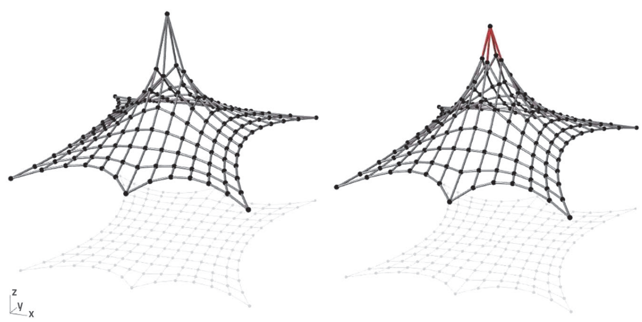


Figure 25. Model 2 - equilibrium position for given force densities, and also for given lengths of marked elements around high anchor

```
def multistepFDM (nodes, elems, supports, qs, fcs = [], lcs = [], steps = 250) :
    qs0 = copy (qs)
    ndof, tdof = table_of_nodal_DsOF (len (nodes), supports)
    nc = _FDM_d (ndof, tdof, nodes, elems, qs0)
    l = list_of_element_lengths (elems, nc)
    f = list_of_element_forces (l, qs0)
    for i in xrange (2, steps + 1) :
        for fj in fcs :
            qs0[fj[0]] = fj[1] / l[fj[0]]
        for lj in lcs :
            qs0[lj[0]] = f[lj[0]] / lj[1]
        nc = _FDM_d (ndof, tdof, nodes, elems, qs0)
        l = list_of_element_lengths (elems, nc)
        f = list_of_element_forces (l, qs0)
    return (nc, f, qs0)
```

Figure 24. Definition of multistepFDM function

The new equilibrium position of nodes, found by using the `multistepFDM` function, with reduced lengths of elements connected to a high anchor (marked in red at Fig. 25 on the right) is closer to a satisfactory solution than the previous one. We say closer, because form finding is a demanding iterative process in which a structurally effective shape needs to be found, that also satisfies different architectural and structural constraints, bearing in mind it's influence on the planned complementary structure as well.

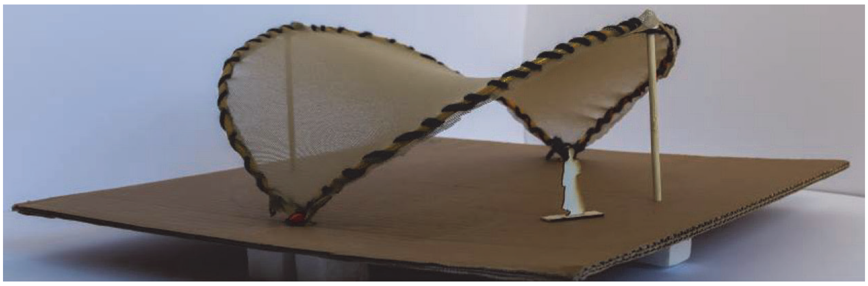


Figure 26. Physical model 3 (authors: S. Omanović, D. Ostrun, T. Pahljina, I. Pavlaković, D. Percan)

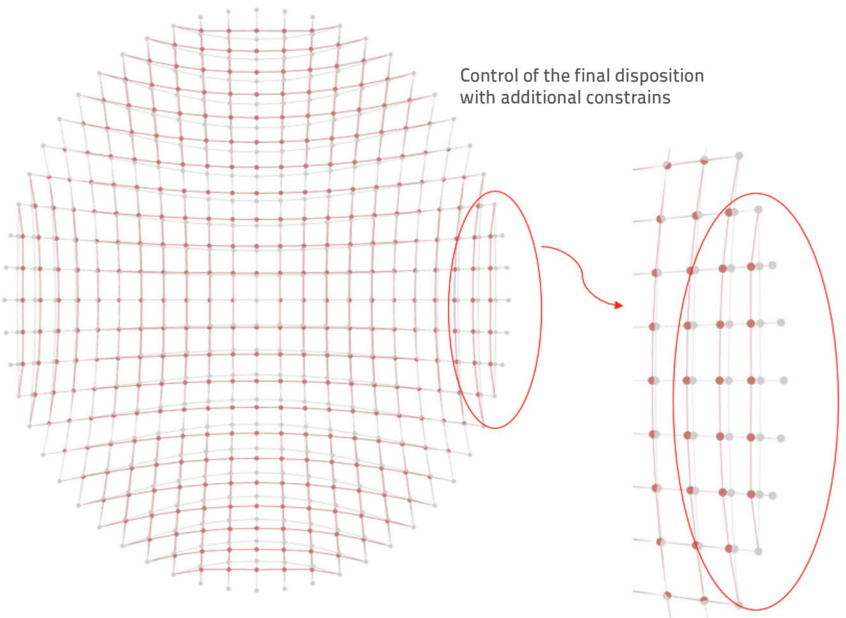


Figure 27. Model 3 - combining unstrained length constraints and length constraints to control distortion of cable net elements (new disposition marked in grey)

In the same way, it is possible to limit the values of forces in individual elements or along the cable. The `multistepFDM` function can also be extended with unstrained length constraints. The introduction of unstrained lengths into the form finding function is shown in [24] and has not been used in classes so far.

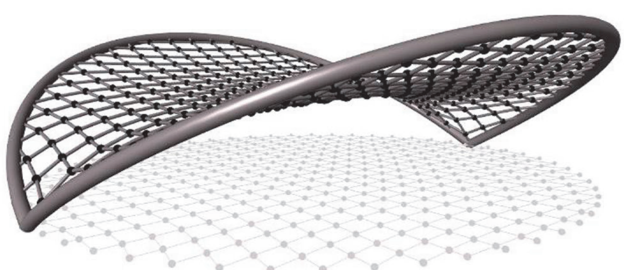


Figure 28. Model 3 – final form finding solution

This type of constrain would be suitable for application, for example, in model 3 (Figure 26) where the net, due to the fixed edge, cannot be controlled by increasing the force densities in the edge cables or setting the desired element lengths. After performing initial form finding (elements and nodes shown in red in Figure 27), the net's shape and disposition can be further controlled by combining length constraints set for the elements connected to the fixed edge together with unstrained length constraints set for the rest of the net. Together these constraints enable control over nets disposition, leading to satisfactory solution marked gray in Figure 27.

Photos: we would like to thank Karlo Jandrić, mag. ing. aedif., for his effort in taking pictures of physical models (Figures 16, 17, 21 and 26).

6. Conclusion and announcement

Instead of a conclusion, we would like to quote the words of a student (also a co-author of the paper) who participated in the course Structural Analysis 2 in the academic year 2019/2020 when physical models were used to teach the students about form finding.

"The creation of a physical model as part of a class encountered general interest and enthusiasm among students. In the absence of practical work at the undergraduate level, the theoretical knowledge was finally given the opportunity to be applied. The interest, motivation and commitment of the students reached a new level, took on a new form. The idea of tensile structures is impressive itself, and how impressive it is to build it with one's own hands (at least on a small scale). The evolution of elastic mesh fabric into a prestressed tensile structure stimulated teamwork and awakened engineers in students, and by that the knowledge acquired in the course got its manifestation. The equilibrium form of the net has become more than 'satisfied independent equilibrium conditions'.

The rules that define the approach towards building a tensile structure are known. They are unambiguous and clear until the moment when the membrane is raised outside the floor plane, when the main problem becomes how to avoid creasing of the fabric. Then changes in support positions, fabric tension and cable tension follow, in order to satisfy all the working principles of tensile structures. Exactly 'manual iteration' was the key to understanding the prestress and negligible flexural stiffness due to which these structures do not have a bending moment or a shear force in the cross-sections. This is abstract until the moment it becomes visible to the eye and accomplished by hand. To pull the fabric to simulate the tensile membrane means to understand why external loads are transmitted by its (pre)tension and to understand the key to the bearing capacity of such structures. Feeling tension under own hand means understanding why a structure is exactly a tensile one.

Figure 29. Building physical models in class Static Analysis 2

Form finding in Civil Engineering classes required motivation, dexterity, resourcefulness, creativity and understanding of the principles of bearing capacity of tensile structures. It made students "feel like engineers" and it stimulated interest in these imposing structures, whose appearance and design process distinguishes from conventional buildings."

The students exhibited their models in the main entrance of the AGG faculty (Figure 30). Since currently teaching is only possible online and, on the other hand, due to positive experience of lecturers and the interest of students in acquiring practical competence, this year the design and static calculation of tensile flexible structures will be taught in a workshop called FormLab, planned to be held in the middle of this year at the Faculty of Civil Engineering in Zagreb as part of the Professional practice at the Faculty of Civil Engineering (a project financed from the European Structural and Investment Funds). One of the goals of the GRASP project is to "raise students' knowledge to a higher level and provide them with practical skills" [25], to which the FormLab workshop will (hopefully) contribute. The workshop will be open to 3rd year undergraduate students and graduate students of the Faculty of Civil Engineering.

Figure 30. Exhibition of student's work in hall of the Faculty (February, 2020)

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